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van Dijk, N.M.

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Nico M. van Dijk

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SIMPLE PERFORMABILITY BOUNDS FOR COMMUNICATION NETWORKS

Nico M. van Dijk

Free University, De Boelelaan 1105, 1081 HV Amsterdam, The Netherlands

A communication network of end-to-end connections is studied with message interruptions such as due to collisions, source interferences or breakdowns. The purpose of the paper is twofold:

(i) To propose a modification approach to obtain simple performance estimates.
(ii) To illustrate how a priori bounds on the accuracy can be concluded.

Two applications are studied:

- A communication network with message collisions such as due to time-slotting.
- A communication network which is subject to total system breakdowns.

For each a simple explicit throughput estimate and error bound on its accuracy is obtained. Further application of the modification approach seems promising.

Keywords Communication network * message collisions * breakdowns * performability estimate * error bound.

1 Introduction

Motivation

Communication networks have become an integral part of present-day organizations and technological developments. Enormous cash-flows are nowadays involved in the design, modeling and evaluation of both to be built and existing systems. Unfortunately, exact analytic expressions for performance measures of interest are usually destroyed by practical features such as message collisions, source interferences or system breakdowns. In well designed systems though the occurrence of such phenomena will be "rare". For quick performance evaluation purposes it thus seems appealing to ignore these features so as to obtain a simple performance estimate.

Results

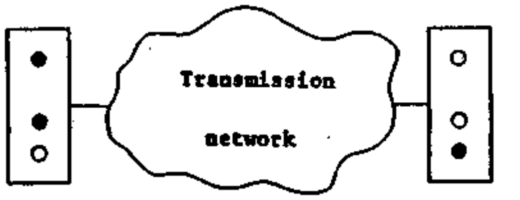
This paper will suggest such estimates for a class of communication networks. In particular, it will study two applications in detail in order to show that formal support by error bounds can be obtained. These applications concern:

- A communication network with message collision probabilities.
- A communication network in which the total system can go down.

For each a simple throughput estimate and explicit error bound of its accuracy will be provided. Further applications such as to study the effect of message retransmissions or propagation delays seem promising.

2 Model and estimates

Consider a communication network of M transmitters (sources), numbered



1,...,M. At any time a source is either in an idle (non-transmitting) or busy (transmitting) mode and the system state is expressed by $H=(h_1,\dots,h_m)$ denoting that sources $h_1,\dots,h_m$ are currently busy. The mode mechanism is determined as follows. When a source h is idle in state H, thus with $h \notin H$, it will request a transmission at an exponential rate:

$$\gamma_h \beta_1(h|H)$$

When a source h is busy in state H, thus with $h \in H$, it will complete its transmission at an exponential rate

$$\mu_h \beta_2(h|H).$$

Here the functions $\beta_1(.|.)$ and $\beta_2(.|.)$ are included to model interference phenomena such as collisions or priorities.
Particular, $\beta_1(h|H) = 0$ is allowed to reflect that state H blocks source h to become busy, for example when source h requires channels already occupied by

busy sources in state H. Similarly, also $\beta_2(h|H) = 0$ is allowed to reflect that an ongoing transmission of source h is interrupted in state H, for example due to a busy source which has higher priority on a common channel or due to a breakdown.

Without loss of generality assume that the underlying continuous-time Markov chain is irreducible at some set S. Clearly, the form of this set is determined by 0-values of the functions β_1 and β_2. Let $\pi(H)$ denote the steady state distribution at S. The following Lemma, adopted from [2], shows when one can conclude an explicit expression for $\pi(H)$.

Lemma 2.1 Suppose that for all $H \in S$:

$$\text{(2.2)} \quad \beta_1(h|H-h) = 0 \Leftrightarrow \beta_2(h|H) = 0$$

$$\text{(2.3)} \quad \beta_1(h|H-h) = \beta_2(h|H) = 1$$

otherwise. Then for all $H \in S$:

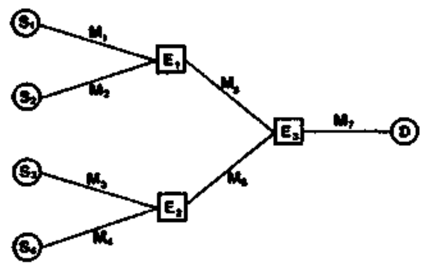
$$\text{(2.4)} \quad \boxed{\pi(H) = c\prod_{h\in H} [\gamma_h/\mu_h].}$$

To illustrate condition (2.2) let us give two examples.

Example 2.1 (Coordinate convex) Assume that $\beta_2(h|H-h) = \beta_2(h|H) = 1$ for all $H \in S$. Then (2.2) is satisfied provided S has the coordinate convex form:

$$\text{(2.5)} \quad H \in S \rightarrow H-h \in S$$

Example: Circuit switching



As an example of (2.5) consider a circuit switched structure with M_i trunks at trunkgroup i and with sources classified in 4-types. A transmission from a source requires a trunk from each trunk group along its end-to-end connection at the same time. Condition (2.5) is then satisfied with S the set of states H in which the numbers of busy type-i sources n_i satisfy the conditions:

$$n_1 \le M_1$$
$$n_1 + n_2 \le M_5$$
$$n_3 + n_4 \le M_6$$
$$n_1 + n_2 + n_3 + n_4 \le M_7$$

In realistic situations, though, also other source interferences may be involved, for example message collisions due to time-slotting, by which condition (2.3) and generally (2.4) will be violated. This will be studied in section 3.1.

Example 2.2 (Priority source/breakdown) Let $\beta_1(.|.) = \beta_2(.|.) = 1$ at S except for states H with source $M \in H$:

$$\text{(2.6)} \quad \beta_1(h|H-h) = \beta_2(h|H) = 0 \qquad (h \in H,\ h \neq M \in H)$$

Source M would thus essentially stop all other sources to work, for example modeling an emergency transmission or a total system breakdown. The conditions (2.2) and (2.3) are then satisfied.

More realistically, though, in such occasions only the $\beta_2(.|.)$ and not the $\beta_1(.|.)$-function would become 0, so that (2.2) and thus also (2.4) fail. This will be studied in section 3.2.

Modification approach

In practice, the conditions (2.2) and (2.3) are usually not perfectly satisfied but with failures occurring only rarely. By slightly modifying the system so as to repair these failures one would thus justify the use of (2.4) as a simple reasonable approximation. Particularly, one can so obtain a simple estimate for the throughput given by:

$$\text{(2.7)} \qquad g = \Sigma_{H\in S}\pi(H)\left[\Sigma_{h\in H}\mu_h\beta_2(h|H)\right].$$

The next section will study two special applications of the modification approach in more detail and present explicit error bounds on the accuracy of the estimates.

3 Two applications

3.1 Message collisions

Consider the system as described in section 2 with S satisfying (2.5) but where for some $\alpha>0$ and all $H, H+h \in S$:

$$\text{(3.1)} \quad \begin{cases} [1-\beta_1(h|H)] \le \alpha \\ [1-\beta_2(h|H)] \le \alpha. \end{cases}$$

Typically, the left hand sides of these inequalities may represent collision probabilities. For example, when system entrances and departures take place in some time-slotted manner with time-slot Δ, we could have:

$$\text{(3.2)} \quad \begin{cases} \beta_1(h|H) = e^{-\Delta\ \Sigma_{R\in H, R\neq H}\gamma_h} \\ \beta_2(h|H) = e^{-\Delta\ \Sigma_{R\in H, R\neq h}\mu_h} \end{cases}$$

representing that only one entrance or departure request can be granted within the same time-slot. Condition (3.1) is satisfied with

$$\text{(3.3)} \quad \boxed{\alpha = \Delta\ \Sigma_h(\gamma_h+\mu_h).}$$

(Estimate)

Consider the modified system with exactly the same state space S satisfying (2.5) and intensities γ_h and μ_h for all h, but with $\beta_2(h|H) = \beta_1(h|H-h) = 1$ for all H∈S, thus as in example 2.1. In words that is, as if the state dependent but small loss probabilities as according to (3.1) are simply ignored. Expression (2.4) for the steady state probabilities $\bar{\pi}(.)$ now applies so that its throughput is easily calculated by

$$\text{(3.4)} \quad \bar{g} = \Sigma_H\ \bar{\pi}(H) \left(\Sigma_{h\in H}\mu_h\right).$$

The following result, which will be partially proven later on, compares the throughput g of the original system and $\bar{g}$ of the modified system as given by (2.7) and (3.4).

Result 3.1

$$\text{(3.5)} \quad \boxed{|g-\bar{g}| \le \alpha\ \Sigma_h(\gamma_h+\mu_h)}$$

Remark Note that the error bound (3.5) is intuitively supported.

3.2 System breakdown

Let $\beta_1(.|.) = \beta_2(.|.) = 1$ at S except for

$$\text{(3.6)} \quad \boxed{\beta_2(h|H) = 0 \text{ for } H \text{ with source } M\in H\ (h\neq M).}$$

(Note the contrast with (2.6)). Source M will thus stop all ongoing transmissions but not the scheduling of new transmissions by idle sources. This corresponds precisely with a so-called independent total system breakdown, which may occur independently of the system state. In that case it is realistic to assume that the downtime fraction is small.

(Estimate)

Consider the above system, but without (3.6). In words that is, source M does not interrupt other sources. Again expression (2.4) applies with π replaced by $\bar{\pi}$ and throughput

$$\text{(3.7)} \quad \bar{g} = \Sigma_{H\in S}\bar{\pi}(H) \left(\Sigma_{h\in H}\mu_h\right)$$

The following result, proven later on, compares the throughput g of the above original system and $\bar{g}$ of this modified system as by (2.7) and (3.7).

Result 3.2

$$\text{(3.8)} \quad \boxed{|g-\bar{g}| \le \left[\frac{\gamma_M}{\gamma_M+\mu_M}\right]\left(\Sigma_h\ \mu_h\right).}$$

Remark 3.2 Note that $\gamma_M/(\gamma_M+\mu_M)$ represents the downtime fraction of the system which should be thought of as a small number like 0.5-2.0%. As the throughput itself is of order $(\Sigma_h\mu_h)$, the relative error of the throughput estimate would thus also be of this small order.

4 Proof

The proof of results 3.1 and 3.2 will follow by applying an approximation theorem from [1]. We will leave the notational transformations and technicalities to the reader, as these are rather straightforward, except for one essential step. This concerns the estimation of the so-called bias-terms (see title of reference [1]) which is needed to apply the theorem. Also, for presentational simplicity we assume here that S has no restrictions, i.e. $\beta(.|.) = \beta(.|.) = 1$.

More precisely, consider the system from example 2.1, that is without source interferences within S. Note that this corresponds to the modified model of application 3.1 as well as application 3.2. With

$$Q = \Sigma_h\left(\gamma_h + \mu_h\right),$$

define functions $V_n(.)$ by $V_0(.) = 0$ and for n>0:

$$\begin{aligned}\text{(4.1)} \quad V_{n+1}(H) &= \Sigma_{h\in H}[\mu_h/Q] \\ &+ \Sigma_{h\notin H}[\gamma_h/Q]\ V_n(H+h) \\ &+ \Sigma_{h\in H}[\mu_h/Q]\ V_n(H-h).\end{aligned}$$

For arbitrary initial stat H the system throughput g is then obtained by

$$\text{(4.2)} \quad g = \lim \frac{Q}{n} V_n(H).$$

Lemma 4.1 For all n and H, H+h ∈ S we have:

$$\text{(4.3)} \quad \boxed{0 \le V_n(H+h)-V_n(H) \le 1.}$$

Proof We will apply induction on n. As $V_0(.) = 0$, (4.3) holds for n = 0. Suppose that (4.3) holds for all n ≤ m, h and H. Then by (4.1), (4.4) $V_{m+1}(H+h) - V_m(H)$

$$=$$

$$\Big\{\Sigma_{\ell\in H+h}[\mu_\ell/Q] + \Sigma_{\ell\notin H+h}[\gamma_\ell/Q]V_m(H+h+\ell) +$$

$$\Sigma_{\ell\in H+h}[\mu_\ell/Q]V_m(H+h-\ell) +$$

$$\Big(1-\Sigma_{\ell\notin H+h}[\gamma_\ell/Q] - \Sigma_{\ell\in H+h}[\mu_\ell/Q]\Big)\ V_m(H+h)\Big\}$$

$$-$$

$$\Big\{\Sigma_{\ell\in H}[\mu_\ell/Q] + \Sigma_{\ell\notin H}[\gamma_\ell/Q]V_m(H+\ell) +$$

$$\Sigma_{\ell\in H}[\mu_\ell/Q]V_m(H-\ell) +$$

$$\Big(1-\Sigma_{\ell\notin H}[\gamma_\ell/Q] - \Sigma_{\ell\in H}[\mu_\ell/Q]\Big)\ V_m(H)\Big\}$$

$$=$$

$$[\mu_h/Q] + \Sigma_{\ell\in H}\ [\mu_\ell/Q]\Big(V_m(H+h-\ell) - V_m(H-\ell)\Big) +$$

$$\Sigma_{\ell\notin H+h}[\gamma_\ell/Q]\ \Big(V_m(H+h+\ell) - V_m(H+\ell)\Big) +$$

$$\Big(1-\Sigma_{\ell\in H+h}[\mu_\ell/Q] - \Sigma_{\ell\notin H+h}[\gamma_\ell/Q]\Big).$$

$$\Big(V_m(H+h) - V_m(H)\Big).$$

By substituting the induction hypothesis (4.3) for $n = m$ and recalling $Q = \Sigma_h(\gamma_h+\mu_h)$ one immediately verifies (4.3) also for $n = m+1$. □

The proof of result 3.1 now completes by theorem 2.1 of [1] and verification its conditions with:

$$\begin{cases} \delta = 0 \\ \Phi(.) = 1 \\ \varepsilon = \alpha\ \Sigma_h\Big(\gamma_h+\mu_h\Big)\ Q^{-1}. \end{cases}$$

The proof of result 3.2 follows similarly by verifying its conditions with:

$$\begin{cases} \delta = 0 \\ \varepsilon = \Sigma_h\mu_h\ Q^{-1} \\ \Phi(H) = 1_{(M\in H)} \\ \beta = \gamma_M/(\gamma_M+\mu_M) \end{cases}$$

Remark By combining the lower estimate 0 from (4.3) with theorem 2.2 of [1] one could also formalize the intuitive obvious result: $g \le \bar{g}$ in both results 3.1 and 3.2.

References

[1] Van Dijk, N.M. (1990), "The importance of bias-terms for error bounds and comparison results", Proceedings of the First International workshop on numerical solutions of Markov chains", Raleigh, 8-10 january 1990, 640-663. To appear: Marcel Dekker (ed. W.J. Stewart).

[2] Van Dijk, N.M. and J.P. Veltkamp (1988), "Product forms for stochastic interference systems", Probability in Engineering and Informational Sciences, Vol.2, 355-376.